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Mathematics

Infinite flag varieties and conjugacy theorems

(Kac–Moody algebra and associated group/integrable representation/Bruhat and Birkhoff decompositions/Schubert variety/Cartan and Borel subalgebras)

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ABSTRACT We study the orbit of a highest-weight vector in an integrable highest-weight module of the group G associated to a Kac–Moody algebra $\mathfrak{g}(A)$. We obtain applications to the geometric structure of the associated flag varieties and to the algebraic structure of $\mathfrak{g}(A)$. In particular, we prove conjugacy theorems for Cartan and Borel subalgebras of $\mathfrak{g}(A)$, so that the Cartan matrix A is an invariant of $\mathfrak{g}(A)$.

1. We first recall some facts about Kac–Moody algebras (see refs. 1 and 2 for details).

A *symmetrizable generalized Cartan matrix* $A = (a_{ij})_{i,j\in I}$ indexed by a nonempty finite set I is a matrix of integers satisfying $a_{ii} = 2$ for all i; $a_{ij} \leq 0$ if $i \neq j$; DA is symmetric for some nondegenerate diagonal matrix D. We fix such a matrix A, assumed for simplicity to be indecomposable.

Fix a base field $\mathbb{F}$ of characteristic zero. Choose a triple $(\mathfrak{h},\Pi,\Pi^\vee)$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbb{F}$ of dimension $|I|$ + corank A and $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*, \Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$.

The *Kac–Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra over $\mathbb{F}$ generated by $\mathfrak{h}$ and symbols e_i and $f_i (i \in I)$ with defining relations: $[\mathfrak{h},\mathfrak{h}] = 0$; $[e_i,f_j] = \delta_{ij}h_i$; $[h,e_i] = \alpha_i(h)e_i$, $[h,f_i] = -\alpha_i(h)f_i$ $(h \in \mathfrak{h})$; $(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j)$ $(i \neq j)$. We have the canonical embedding $\mathfrak{h} \subset \mathfrak{g}$ and linearly independent *Chevalley generators* $e_i, f_i (i \in I)$ for the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$. The center $\mathfrak{c}$ of $\mathfrak{g}$ lies in $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{g}' = \Sigma\, \mathbb{F}h_i$. Every ideal of $\mathfrak{g}$ contains $\mathfrak{g}'$ or is contained in $\mathfrak{c}$ (3).

Define an involution ω of $\mathfrak{g}$ by requiring $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, and $\omega(h) = -h (h \in \mathfrak{h})$. Let $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ generated by the $e_i (i \in I)$, and put $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+, \mathfrak{n}_- = \omega(\mathfrak{n}_+)$, $\mathfrak{b}_- = \omega(\mathfrak{b}_+)$. We have the vector space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

We have the *root space decomposition* $\mathfrak{g} = \oplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Put $Q = \Sigma_{i\in I}\mathbb{Z}\alpha_i$, $Q_+ = \Sigma_{i\in I}\mathbb{Z}_+\alpha_i$ (where $\mathbb{Z}_+ = \{0,1,\ldots\}$), and define a partial order on $\mathfrak{h}^*$ by: $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. A *root* (resp. *positive root*) is an element of $\Delta := \{\alpha \in \mathfrak{h}^* | \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$ (resp. $\Delta_+ := \Delta \cap Q_+$). We have: $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}_\pm = \oplus_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}$.

Let Aut(A) be the group of all permutations σ of I satisfying $a_{\sigma(i),\sigma(j)} = a_{ij}$. We regard Aut$(A)$ as a subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{g}')$ by requiring $\sigma(e_i) = e_{\sigma(i)}$, $\sigma(f_i) = f_{\sigma(i)}$. We define the *outer automorphism group* Out(A) of $\mathfrak{g}'$ to be Aut(A) if $\dim \mathfrak{g}' < \infty$ and $\{1,\omega\} \times \mathrm{Aut}(A)$ otherwise.

Define $r_i \in \mathrm{Aut}_{\mathbb{F}}(\mathfrak{h})$, $i \in I$, by $r_i(h) = h - \alpha_i(h)h_i$, and put $S = \{r_i | i \in I\}$. S generates the *Weyl group* W, and (W,S) is a Coxeter system. W preserves the root system Δ. A *real* (resp. *imaginary*) root is an element of $\Delta^{\mathrm{re}} := W(\Pi)$ (resp. $\Delta^{\mathrm{im}} := \Delta\backslash\Delta^{\mathrm{re}}$). If $\alpha \in \Delta^{\mathrm{re}}$, then $\dim \mathfrak{g}_\alpha = 1$ and $\Delta \cap \mathbb{Z}\alpha = \{\alpha,-\alpha\}$; if $\alpha \in \Delta^{\mathrm{im}}$, then $\mathbb{Z}\alpha \subset \Delta \cup \{0\}$. Put $\Delta^{\mathrm{re}}_+ = \Delta^{\mathrm{re}} \cap \Delta_+$. For $\alpha \in \Delta^{\mathrm{re}}$, write $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in I$; then $\alpha^\vee := w(h_i)$ and $r_\alpha := wr_iw^{-1}$ depend only on α.

We choose a nondegenerate $\mathfrak{g}$-invariant symmetric $\mathbb{F}$-bilinear form $(.|.)$ on $\mathfrak{g}$ such that $(h_i|h_i)$ is positive rational for all $i \in I$. $(.|.)$ is nondegenerate and W-invariant on $\mathfrak{h}$ and, hence, induces a form $(.|.)$ on $\mathfrak{h}^*$. One has $(\alpha|\alpha) > 0$ if $\alpha \in \Delta^{\mathrm{re}}$. $(.|.)$ induces a nondegenerate form on $\mathfrak{g}'/\mathfrak{c}$, which is unique up to multiples.

2. We now construct the group G associated to the Lie algebra $\mathfrak{g}'$. Other approaches may be found in refs. 4–8.

A $\mathfrak{g}'$-module V, or (V,π), where $\pi: \mathfrak{g}' \to \mathrm{End}_{\mathbb{F}}(V)$, is called *integrable* if $\pi(e)$ is locally nilpotent whenever $e \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^{\mathrm{re}}$ $(\alpha \in \pm\Pi$ suffices). $(\mathfrak{g},\mathrm{ad})$ is an integrable $\mathfrak{g}'$-module.

Let G^* be the free product of the additive groups $\mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$, with canonical inclusions $i_\alpha: \mathfrak{g}_\alpha \to G^*$. For any integrable $\mathfrak{g}'$-module (V,π), define a homomorphism $\pi^*: G^* \to \mathrm{Aut}_{\mathbb{F}}(V)$ by $\pi^*(i_\alpha(e)) = \exp \pi(e)$. Let N^* be the intersection of all $\mathrm{Ker}(\pi^*)$, put $G = G^*/N^*$, and let $q: G^* \to G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha (\alpha \in \Delta^{\mathrm{re}})$, put $\exp e = q(i_\alpha(e))$, so that $U_\alpha := \exp \mathfrak{g}_\alpha$ is an additive one-parameter subgroup of G. The $U_\alpha (\alpha \in \pm\Pi)$ generate G, and G is its own derived group. Denote by U_+ (resp. U_-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$), $\alpha \in \Delta^{\mathrm{re}}_+$.

Example: Let A be the Cartan matrix of a split simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{F}$. Then the group G associated to $\mathfrak{g} \cong \mathfrak{g}'(A)$ is the group $\underset{\sim}{G}(\mathbb{F})$ of $\mathbb{F}$-valued points of the connected simply-connected algebraic group $\underset{\sim}{G}$ associated to $\mathfrak{g}$, and $U_+ = \underset{\sim}{U}(\mathbb{F})$ for some maximal unipotent subgroup $\underset{\sim}{U}$ of $\underset{\sim}{G}$. Now let $\tilde{A}$ be the extended Cartan matrix of $\mathfrak{g}$. Then the group G associated to $\mathfrak{g}'(\tilde{A})$ is a central extension by $\mathbb{F}^*$ of $\underset{\sim}{G}(\mathbb{F}[z,z^{-1}])$, and $U_+ \cong \{g(z) \in \underset{\sim}{G}(\mathbb{F}[z]) | g(0) \in \underset{\sim}{U}(\mathbb{F})\}$. In particular, if $\underset{\sim}{G} = SL_2$, then U_+ is isomorphic to the free product of the abelian groups

$$\begin{pmatrix} 1 & \mathbb{F}[z] \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ z\mathbb{F}[z] & 1 \end{pmatrix},$$

so that no proper subset of $\{U_\alpha | \alpha \in \Delta^{\mathrm{re}}_+\}$ generates U_+.

To any integrable $\mathfrak{g}'$-module (V,π) we associate the homomorphism (again denoted by) $\pi: G \to \mathrm{Aut}_{\mathbb{F}}(V)$ satisfying $\pi(\exp e) = \exp \pi(e)$ for $e \in \mathfrak{g}_\alpha (\alpha \in \Delta^{\mathrm{re}})$. The homomorphism associated to $(\mathfrak{g},\mathrm{ad})$, denoted Ad, maps G into $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{g})$. The kernel of Ad is the center C of G, and Ad(G) acts faithfully on $\mathfrak{g}'/\mathfrak{c}$. We have $\pi(\mathrm{Ad}(g)x) = \pi(g)\pi(x)\pi(g)^{-1}$ for any integrable $\mathfrak{g}'$-module (V,π) and all $g \in G$, $x \in \mathfrak{g}'$. It follows that if (V,π) is an integrable $\mathfrak{g}'$-module with Ker $\pi \subset \mathfrak{c}$, then (on G) Ker $\pi \subset C$.

For each $i \in I$ we have a unique homomorphism $\phi_i: SL_2(\mathbb{F}) \to G$ satisfying:

$$\phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp te_i, \quad \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tf_i \; (t \in \mathbb{F}).$$

Let $G_i = \phi_i(SL_2(\mathbb{F}))$, $H_i = \phi_i(\{\mathrm{diag}(t,t^{-1}) | t \in \mathbb{F}^*\})$, and let N_i be the normalizer of H_i in G_i. Let H (resp. N) be the subgroup of G generated by the H_i (resp. N_i); H is an abelian normal subgroup

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of N. Using the modules $L(\Lambda)$ (see section 3), we show that the ϕ_i are monomorphisms and that H is the direct product of the H_i. Using the fact that W is a Coxeter group, we have an isomorphism $\phi: W \to N/H$ such that $\phi(r_i)$ is the coset $N_iH\backslash H$. We identify W and N/H using ϕ; this gives sense to expressions such as wH and wU_+w^{-1} occurring in the sequel. We put $B_+ = HU_+$, $B_- = HU_-$.

LEMMA 1. *If* $\alpha,\beta \in \Delta^{re}$ *and* $\alpha(\beta^\vee) \geq 0$, *then:*

(a) $(\mathbb{Z}_+\alpha + \mathbb{Z}_+\beta) \cap \Delta = \{\alpha,\beta,\alpha+\beta\} \cap \Delta^{re}$.

(b) $(U_\alpha, U_\beta) \subset U_{\alpha+\beta}$ *if* $\alpha + \beta \in \Delta^{re}$ *and* $= \{1\}$ *otherwise.*

Proof: If $\gamma = m\alpha + n\beta \in \Delta$, where $m,n \in \mathbb{Z}_+$, then $\gamma(\beta^\vee) \geq n\beta(\beta^\vee) = 2n$, so that by using root strings, $m\alpha \in \Delta \cup \{0\}$ and, hence, $m \leq 1$. Similarly, $n \leq 1$. In particular, $2\gamma \notin \Delta$, so that $\gamma \in \Delta^{re}$. This proves part *a*; part *b* follows from part *a*.

COROLLARY 1 (cf. ref. 8). *Let* $\alpha \in \Pi$ *and let* $U^\alpha = \langle xyx^{-1} | x \in U_\alpha, y \in U_\beta$ *for some* $\beta \in \Delta_+^{re}\backslash\{\alpha\}\rangle$. *Then* $U^\alpha = U_+ \cap r_\alpha U_+ r_\alpha^{-1}$ *and* $U_+ = U_\alpha \propto U^\alpha$.

Proof: The inclusion $r_\alpha x U_\beta x^{-1} r_\alpha^{-1} \subset U^\alpha$ for $\beta \in \Delta_+^{re}\backslash\{\alpha\}$ and $x \in U_\alpha$ follows from *Lemma 1* if $\alpha(\beta^\vee) \geq 0$; the case $\alpha(\beta^\vee) < 0$ reduces to the first one.

COROLLARY 2.

(a) (G, B_+, N, S) *is a Tits system.*

(b) $G = \bigcup_{w \in W} B_+ w B_+$ (*Bruhat decomposition*).

(c) $G = \bigcup_{w \in W} B_- w B_+$ (*Birkhoff decomposition*).

Proof: The proofs of parts *a* and *b* are standard using *Corollary 1* (cf. ref. 9). For part *c* one checks that

$$B_- w B_+ r_i \subset B_- w B_+ \cup B_- w r_i B_+.$$

The proof of the following lemma is not difficult; it involves a completion of the enveloping algebra of $\mathfrak{n}_+$.

LEMMA 2.

(a) Ad *is faithful on* U_+.

(b) *Let* $w \in W$. *Then to any* $e \in \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ *there corresponds a* $u \in U_+$ *such that for any integrable* $\mathfrak{g}'$*-module* (V,π), $\pi(e)$ *is locally nilpotent and* $\pi(u) = \exp \pi(e)$.

COROLLARY 3.

(a) $U_+ \cap B_- = \{1\}$.

(b) $C \subset H$.

(c) $C \cong \mathrm{Hom}(\mathbb{Z}^I/A\mathbb{Z}^I, \mathbb{F}^*)$.

For the next corollary, we take $\mathbb{F} = \mathbb{C}$. Define a conjugate-linear involution ω_0 of $\mathfrak{g}'$ by requiring $\omega_0(e_i) = -f_i, \omega_0(f_i) = -e_i (i \in I)$, and let K be the fixed-point set of the corresponding involution of G. For $i \in I$, put $K_i = G_i \cap K \cong SU_2$, $H_i^+ = \phi_i(\{\mathrm{diag}(t,t^{-1}) | t \in \mathbb{R}, t > 0\})$, $H_+ = \Pi_{i\in I} H_i^+$. Using *Corollary 2*, the argument in ref. 9, section 8, proves *Corollary 4*.

COROLLARY 4.

(a) K *is generated by the* $K_i (i \in I)$.

(b) (*Iwasawa decomposition*) *The map* $\phi: K \times H_+ \times U_+ \to G$ *defined by* $\phi(k,h,u) = khu$ *is a bijection.*

For $X \subset \Pi$, let W_X be the subgroup of W generated by $\{r_\alpha | \alpha \in X\}$, and put $P_X = B_+W_XB_+$. P_X and $\omega(P_X)$ are called *standard parabolic subgroups* of G (their properties in the framework of Tits systems can be found in ref. 10). Let U^X be the smallest normal subgroup of U_+ containing U_β for all $\beta \in \Delta_+^{re}\backslash\mathbb{Z}X$, let M_X be the subgroup of G generated by H and the U_β $(\beta \in \pm X)$, and let $U_X = U_+ \cap M_X$.

COROLLARY 5. *Let* X *be a subset of* Π. *Then:*

(a) $P_X = M_X \propto U^X$; $M_X = P_X \cap \omega(P_X)$;

$$U^X = \bigcap_{w \in W_X} wU_+w^{-1}.$$

In particular,

$$\bigcap_{w \in W} wU_+w^{-1} = \{1\}.$$

(b) $U^X = (U^X \cap (wU_+w^{-1}))(U^X \cap (wU_-w^{-1}))$ *for every* $w \in W$.

(c) *For every* $g \in G$ *there exists a unique* $(u,n,u') \in U_+ \times N \times U_+$ (*resp.* $U_- \times N \times U_+$) *such that* $u \in nU_-n^{-1}$ *and* $g = unu'$.

COROLLARY 6. N (*resp.* H) *is the normalizer* (*resp. centralizer*) *of* H *in* G.

Proof: Because $G = NU_-U_+$ by *Corollary 5c*, it suffices to show that the normalizer of H in U_- (resp. U_+) is trivial. This follows from *Corollary 1* and the last part of *Corollary 5a*.

3. Now we turn to representation theory. Put $P = \{\Lambda \in \mathfrak{h}^* | \Lambda(h_i) \in \mathbb{Z} (i \in I)\}$, P_+(resp. P_{++}) $= \{\Lambda \in P | \Lambda(h_i) \geq 0$ (resp. > 0), all $i \in I\}$.

We fix in sections 3 and 4 a $\Lambda \in P_+$. Then there exists an irreducible $\mathfrak{g}$-module $(L(\Lambda), \pi_\Lambda)$, unique up to isomorphism, containing a $v^+ \neq 0$ satisfying: $\pi_\Lambda(\mathfrak{n}_+)v^+ = (0)$; $\pi_\Lambda(h)v^+ = \Lambda(h)v^+$ $(h \in \mathfrak{h})$. $L(\Lambda)$ is an absolutely irreducible integrable $\mathfrak{g}'$-module, and we have $L(\Lambda) = \pi_\Lambda(U(\mathfrak{n}_-))v^+$, $\mathrm{End}_{\mathfrak{g}'}(L(\Lambda)) = \mathbb{F}I_{L(\Lambda)}$. The module $L(\Lambda)$ is called an *integrable module with highest weight* Λ (see refs. 1 and 2 for details).

We have the *weight space decomposition* $L(\Lambda) = \oplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda$, where $L(\Lambda)_\lambda = \{v \in L(\Lambda) | h(v) = \lambda(h)v$ for all $h \in \mathfrak{h}\}$. Elements of $P(\Lambda) := \{\lambda \in \mathfrak{h}^* | L(\Lambda)_\lambda \neq (0)\}$ are called *weights* of $L(\Lambda)$. We have $\mathbb{F}v^+ = L(\Lambda)_\Lambda = \{v \in L(\Lambda) | \mathfrak{n}_+(v) = (0)\}$; elements of $\mathbb{F}^*v^+$ are called *highest-weight vectors*. We have $P(\Lambda) \subset \Lambda - Q_+$, and $\dim L(\Lambda)_{w(\lambda)} = \dim L(\Lambda)_\lambda$ if $\lambda \in \mathfrak{h}^*$, $w \in W$.

Regarded as a $\mathfrak{g}$-module under $\pi_\Lambda^* := \pi_\Lambda \circ \omega$, $L(\Lambda)$ is denoted $L^*(\Lambda)$, and v^+ is denoted v^-. Let $\langle\, , \rangle$ be the unique $\mathfrak{g}$-invariant $\mathbb{F}$-bilinear form on $L(\Lambda) \times L^*(\Lambda)$ satisfying $\langle v^+, v^- \rangle = 1$; it is non-degenerate.

By using *Corollary 3b*, it is easy to see that the actions of $\mathfrak{g}'$ and G on the direct sum of all integrable modules with fundamental highest weights are faithful.

Let $X = \{\alpha \in \Pi | \Lambda(\alpha^\vee) = 0\}$. Then *Corollaries 2b* and *5a* imply *Corollary 7*.

COROLLARY 7. P_X (*resp.* M_X) *is the stabilizer of* $\mathbb{F}v^+$ (*resp.* $v^+ \otimes v^-$) *in* G.

Some time ago, B. Kostant proved (cf. ref. 11) that the ideal of the orbit of a highest-weight vector in a nontrivial irreducible finite-dimensional representation of a simple Lie group over $\mathbb{C}$ is generated by quadratic polynomials. His result and proof can be extended to the module $L(\Lambda)$ (19). We need only the following explicit form of the equations for $G(v^+)$.

LEMMA 3. *For each* $\alpha \in \Delta \cup \{0\}$, *choose dual bases* $\{x_\alpha^{(i)}\}$ *of* $\mathfrak{g}_\alpha$ *and* $\{y_\alpha^{(i)}\}$ *of* $\mathfrak{g}_{-\alpha}$. *Then all* v *from the orbit* $G(v^+)$ *satisfy*

$$(\Lambda|\Lambda)\, v \otimes v = \sum_{\alpha \in \Delta \cup \{0\}} x_\alpha^{(i)}(v) \otimes y_\alpha^{(i)}(v). \qquad (*)$$

Proof: $v \otimes v$ lies in the highest component $V \cong L(2\Lambda)$ of $L(\Lambda) \otimes L(\Lambda)$ because $v^+ \otimes v^+$ does. The equation (*) of *Lemma 3* now follows from the fact that the generalized Casimir operator (see refs. 1 and 2 for a definition) acts as (known) scalars on $L(\Lambda)$ and $L(2\Lambda)$.

Let $\mathcal{V}$ be the set of all non-zero $v \in L(\Lambda)$ satisfying Eq. (*). It is a G-invariant cone. For $v \in L(\Lambda)$, write $v = \Sigma_\lambda v_\lambda$, where $v_\lambda \in L(\Lambda)_\lambda$. Put $\mathrm{supp}(v) = \{\lambda \in P(\Lambda) | v_\lambda \neq 0\}$, and let $S(v)$ be the convex hull of supp(v). We now prove a crucial lemma.

LEMMA 4. *If* $v \in \mathcal{V}$, *then:*

(a) *The vertices of the polyhedron* S(v) *lie in the orbit* W(Λ).

(b) *The edges of* S(v) *are parallel to real roots.*

The lemma follows from a simple observation: Let $\mathfrak{g} = \oplus_{r\in\mathbb{R}} \mathfrak{g}_r$ be an $\mathbb{R}$-gradation of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{h}$, and let $L(\Lambda) = \oplus_{r\in\mathbb{R}} L(\Lambda)_r$ be the corresponding gradation of $L(\Lambda)$ with $v^+ \in L(\Lambda)_0$. Then: (*i*) If $v \in \mathcal{V} \cap L(\Lambda)_r$, then $v \in L(\Lambda)_{w(\Lambda)}$ for some $w \in W$. (*ii*) The highest component of any $v \in \mathcal{V}$ lies in $\mathcal{V}$.

Indeed, aside from "real" in part *b*, *Lemma 4* is immediate from the observation. Because the difference of two elements of $W(\Lambda)$ cannot be a non-zero multiple of an imaginary root, *Lemma 4* follows. To prove the observation, note that if $\lambda, \mu \in P(\Lambda)$, then $(\Lambda|\Lambda) - (\lambda|\mu) \geq 0$, with equality iff $\lambda = \mu \in W(\Lambda)$ (2). By examining the $L(\Lambda)_r \otimes L(\Lambda)_r$ component of Eq. (*), *i* follows. Similarly, *ii* follows by considering the highest component of Eq. (*) in the $\mathbb{R}$-gradation of $L(\Lambda) \otimes L(\Lambda)$ defined by

$$L(\Lambda)_r \otimes L(\Lambda)_s \subset (L(\Lambda) \otimes L(\Lambda))_{r+s}.$$

4. Now we turn to the study of the cone $\mathcal{V}$. Put $\tilde{H} = \mathrm{Hom}(Q, \mathbb{F}^*)$, and define a homomorphism Ad: $\tilde{H} \to \mathrm{Aut}_{\mathbb{F}}(\mathfrak{g})$ by $\mathrm{Ad}(h)x = h(\alpha)x$ if $x \in \mathfrak{g}_\alpha$. Ad induces an action of $\tilde{H}$ on G, defining $\tilde{H} \propto G$, to which Ad extends in the obvious way. We extend the action of G on $L(\Lambda)$ to $\tilde{H} \propto G$ by requiring $\tilde{H}$ to fix v^+.

Define the *Bruhat order* $\succcurlyeq$ on the orbit $W(\Lambda)$ to be the partial order generated by: $\lambda \succcurlyeq \mu$ if $\lambda \geq \mu$ and $\lambda = r_\alpha(\mu)$ for some $\alpha \in \Delta^{\mathrm{re}}$. Given $\lambda \in W(\Lambda)$, we put:

$$\mathcal{V}(\lambda)_+ = \{v \in \mathcal{V} | \mathrm{supp}(v) \geq \lambda \quad \text{and} \quad \lambda \in \mathrm{supp}(v)\};$$

$$\mathcal{V}(\lambda)_- = \{v \in \mathcal{V} | \mathrm{supp}(v) \leq \lambda \quad \text{and} \quad \lambda \in \mathrm{supp}(v)\};$$

$$\overline{\mathcal{V}(\lambda)_+} = \bigcup_{\mu \succcurlyeq \lambda} \mathcal{V}(\mu)_+; \overline{\mathcal{V}(\lambda)_-} = \bigcup_{\lambda \succcurlyeq \mu} \mathcal{V}(\mu)_-;$$

$$U(\lambda) = \bigcap_{w \in W: w(\Lambda) = \lambda} wU_-w^{-1}; U_\pm(\lambda) = U(\lambda) \cap U_\pm;$$

$$\Phi(\lambda) = \{\alpha \in \Delta_+^{\mathrm{re}} | \mathrm{r}_\alpha(\lambda) > \lambda\}.$$

Note that $U_+(\lambda)$ is generated by the U_β ($\beta \in \Phi(\lambda)$).

THEOREM 1. *Let* ε *be* + *or* −. *Then:*

(a) $\mathcal{V}$ *is the disjoint union of the* $\mathcal{V}(\lambda)_\varepsilon$, $\lambda \in \mathrm{W}(\Lambda)$.

(b) *If* $\lambda \in \mathrm{W}(\Lambda)$, *then the group* $\mathbb{F}^* \times \mathrm{U}_\varepsilon(\lambda)$ *acts simply-transitively on* $\mathcal{V}(\lambda)_\varepsilon$. *In particular,* $\mathbb{F}^* \times \mathrm{G}$ *acts transitively on* $\mathcal{V}$.

(c) *Let* $\mathcal{T}$ *be a* U_ε*-invariant topology on* $\mathcal{V}$ *such that for all* $\lambda \in \mathrm{W}(\Lambda)$: (i) $\{v \in \mathcal{V} | v_\lambda \neq 0\}$ *is* $\mathcal{T}$*-open;* (ii) v_λ *lies in the* $\mathcal{T}$*-closure of* $\mathbb{F}^*\tilde{\mathrm{H}}(v)$ *if* $v \in \mathcal{V}$ *and* $v_\lambda \neq 0$. *Then for every* $\lambda \in \mathrm{W}(\Lambda)$, *the set* $\overline{\mathcal{V}(\lambda)_\varepsilon}$ *is the* $\mathcal{T}$*-closure of* $\mathcal{V}(\lambda)_\varepsilon$.

Proof. Part *a* is clear by *Lemma 4a*. Part *b* is proved as follows, by "killing off" edges of S(v).

Fix $\lambda \in W(\Lambda)$. If $v = \Sigma_\mu v_\mu \in \mathcal{V}$ and $v_\lambda \neq 0$, we put $\Phi'(v) = \{\alpha \in \Phi(\lambda) | [\lambda, r_\alpha(\lambda)]$ is an edge of $S(v)\}$. $\Phi(v) = \Phi(\lambda) \cap \mathbb{Q}(\lambda - S(v))$. Then we have:

(*i*) $\Phi'(v) \subset \Phi(v)$; $v \in \mathcal{V}(\lambda)_-$ if $\Phi'(v) = \emptyset$.

(*ii*) If $\alpha \in \Phi'(v)$, then there exists $e \in \mathfrak{g}_\alpha$ such that

$$\alpha \notin \Phi((\exp e)v) \subset \Phi(v).$$

Indeed, *i* is immediate from *Lemma 4*. To prove *ii*, choose non-zero $e' \in \mathfrak{g}_\alpha$. Because $\mathbb{F}\pi_\Lambda(e')^{-\lambda(\alpha^\vee)}v_\lambda = L(\Lambda)_{r_\alpha(\lambda)}$, it is easy to see that there exists t in an algebraic closure of $\mathbb{F}$ such that $r_\alpha(\lambda) \notin \mathrm{supp}((\exp te')v)$. Then $\alpha \notin \Phi$ ((exp te')v), by *Lemma 4;* in particular, $\lambda + \alpha \notin \mathrm{supp}((\exp te')v)$, which implies $t \in \mathbb{F}$. Taking $e = te'$, *ii* holds. *i* and *ii* imply the following.

(*iii*) If $v \in \mathcal{V}$ and $v_\lambda \neq 0$, then there exists $u \in U_+(\lambda)$ such that $\pi_\Lambda(u)v \in \mathcal{V}(\lambda)_-$.

In particular, taking $v \in \mathcal{V}(\lambda)_+$ in *iii*, we obtain:

$$\pi_\Lambda(u)v \in \mathcal{V}(\lambda)_+ \cap \mathcal{V}(\lambda)_- = L(\Lambda)_\lambda \backslash \{0\} \subset \mathbb{F}^* N(v^+).$$

By varying λ, this and part *a* imply:

(*iv*) $\mathcal{V} = \mathbb{F}^* G(v^+)$.

From *iv* and *Corollary 2c* we deduce:

(*v*) $\mathcal{V}(\Lambda)_- = \mathbb{F}^* U_-(v^+)$.

v and *Corollaries 5a* and *7* imply:

(*vi*) $\mathbb{F}^* \times U(\Lambda)$ acts simply transitively on $\mathcal{V}(\Lambda)_-$.

From *vi* and *Corollary 5b*, *Theorem 1b* follows.

Finally, we prove *Theorem 1c*. First, let $\varepsilon = +$. Fix $\lambda \in W(\Lambda)$, and let $Y(\lambda)$ be the $\mathcal{T}$-closure of $\mathcal{V}(\lambda)_+$. $Y(\lambda)$ is U_+-invariant because $\mathcal{T}$ and $\mathcal{V}(\lambda)_+$ are. If $v \in \mathcal{V}(\lambda)_+$ and μ is a vertex of $S(v)$, then $\mu \succcurlyeq \lambda$ by *Lemma 4*. By *i* in *Theorem 1c* the same holds if $v \in Y(\lambda)$, so that $Y(\lambda) \subset \overline{\mathcal{V}(\lambda)_+}$ by part *a*. To prove the reverse inclusion, it suffices to show that if $\alpha \in \Delta_+^{\mathrm{re}}$ and $r_\alpha(\lambda) > \lambda$, then $\mathcal{V}(r_\alpha(\lambda))_+ \subset Y(\lambda)$. For this, note that $\pi_\Lambda(\exp \mathfrak{g}_\alpha)(L(\Lambda)_\lambda \backslash \{0\}) \subset \mathcal{V}(\lambda)_+$, so that $L(\Lambda)_{r_\alpha(\lambda)} \backslash \{0\} \subset Y(\lambda)$ by *ii* in *Theorem 1c*. Because $Y(\lambda)$ is U_+-invariant, we obtain $\mathcal{V}(r_\alpha(\lambda))_+ \subset Y(\lambda)$ from *Theorem 1b*. Similarly, *Theorem 1c* holds for $\varepsilon = -$.

5. Now we apply our results to the structure theory of the Kac–Moody algebra $\mathfrak{g}$.

LEMMA 5.

(a) *Let* $\Lambda \in \mathrm{P}_+$ *and let* $\mathfrak{m}$ *be a subalgebra of* $\mathfrak{n}_+$ *of finite codimension. Then* $\mathrm{L}(\Lambda)^{\mathfrak{m}}$: = $\{v \in \mathrm{L}(\Lambda) | \pi_\Lambda(\mathfrak{m})v = (0)\}$ *is finite-dimensional.*

(b) *If* $\Lambda \in \mathrm{P}_{++}$, *then* $\mathfrak{b}_+$ *is the stabilizer of* $\mathbb{F}v_\Lambda$ *in* $\mathrm{L}(\Lambda)$.

Proof. Let $\Lambda \in P_+$. Because Y: = $\cup_{i\in I} \mathrm{Ad}(U_+)e_i$ spans $\mathfrak{n}_+$, there exist $y_1, \ldots, y_r \in Y$ such that $\mathfrak{n}_+ = \mathfrak{m} + \mathbb{F}y_1 + \ldots + \mathbb{F}y_r$. V: = $\pi_\Lambda^*(U(\mathbb{F}y_1) \ldots U(\mathbb{F}y_r))v^-$ is finite-dimensional because the y_j are locally nilpotent on $L^*(\Lambda)$. The linear map from $L(\Lambda)^{\mathfrak{m}}$ to V^* induced by $\langle\ ,\ \rangle$ is injective because $\mathrm{L}^*(\Lambda) = V + \pi_\Lambda^*(\mathfrak{m})L^*(\Lambda)$. This proves part *a*.

Let $\Lambda \in P_{++}$. Because $\pi_\Lambda(f_i)v^+ \neq 0$ for all $i \in I$, it is clear that V: = $\{x \in \mathfrak{n}_- | \pi_\Lambda(x)v^+ = 0\}$ is ad($\mathfrak{b}_+$)-stable. The ideal ad($U(\mathfrak{n}_-)$)V of $\mathfrak{g}$ is contained in $\mathfrak{n}_-$ and, hence, is (0). This proves part *b*.

Let $\mathfrak{p}$ be a Lie algebra and (V, π) a $\mathfrak{p}$-module, both over $\mathbb{F}$. We call $\mathfrak{p}$: (*i*) π*-finite* if $\pi(U(\mathfrak{p}))v$ is finite-dimensional for all $v \in V$; (*ii*) π*-triangular* if for any $v \in V$ there exist $\pi(\mathfrak{p})$-stable subspaces $V_0 \subset \ldots \subset V_n$ of V with $v \in V_n$ and dim $V_j = j$, $0 \leq j \leq n$; and (*iii*) π*-diagonalizable* (resp. π*-semisimple*) if V is a sum of one-dimensional (resp. finite-dimensional irreducible) $\mathfrak{p}$-submodules. We call $x \in \mathfrak{p}$ π-finite, -triangular, etc., if $\mathbb{F}x$ is.

LEMMA 6. *The following are equivalent for a subalgebra*' $\mathfrak{p}$ *of* $\mathfrak{g}$:

(a) $\mathfrak{p}$ *is* $\mathrm{ad}_{\mathfrak{g}'}$*-triangular.*

(b) $\mathfrak{p}$ *is* π_Λ*- and* π_Λ^**-triangular for all* $\Lambda \in \mathrm{P}_+$.

(c) $\mathrm{Ad}(\mathrm{g})\mathfrak{p} \subset \mathfrak{b}_+ \cap \mathrm{w}(\mathfrak{b}_-)$ *for some* $\mathrm{g} \in \mathrm{G}$ *and* $\mathrm{w} \in \mathrm{W}$.

Proof: Clearly, *c* implies *a*. We will show that *a* implies *b* and that *b* implies *c*. We assume $\mathfrak{p}$ to be finite-dimensional and solvable because both *a* and *b* imply this.

Assume *a*. Because $\mathfrak{p}$ is finite-dimensional, there exists a finite-dimensional subspace A of $\mathfrak{g}'$ such that $\mathfrak{b}_+ + [\mathfrak{p}, \mathfrak{b}_+] = \mathfrak{b}_+ + A$. Put $V = \mathrm{ad}(U(\mathfrak{p}))\mathfrak{b}_+$. Then $V/\mathfrak{b}_+$ is finite-dimensional because $\mathfrak{p}$ is $\mathrm{ad}_{\mathfrak{g}'}$-finite and $\mathrm{V} = \mathfrak{b}_+ + \mathrm{ad}(U(\mathfrak{p}))\mathrm{A}$. Hence, the $\mathfrak{p}$-stable subalgebra $\mathfrak{m}$: = $\{x \in \mathfrak{g} | (x|V) = (0)\}$ of $\mathfrak{n}_+$ is of finite codimension. Let $\Lambda \in P_+$. Then $L(\Lambda)^{\mathfrak{m}}$ is $\pi_\Lambda(\mathfrak{p})$-stable, and is finite-dimensional by *Lemma 5a*. $\mathfrak{p}$ is π_Λ-finite because $\mathfrak{p}$ is $\mathrm{ad}_{\mathfrak{g}'}$-finite and $v^+ \in L(\Lambda)^{\mathfrak{m}}$ is a cyclic vector for $L(\Lambda)$ under

$\mathfrak{g}'$. If $x \in \mathfrak{p}$, then the eigenvalues of $\mathrm{ad}_{\mathfrak{g}'}(x)$ lie in $\mathbb{F}$; because $L(\Lambda)$ is an absolutely irreducible $\mathfrak{g}'$-module, it follows that all eigenvalues of $\pi_\Lambda(x)$ are congruent modulo $\mathbb{F}$ and, hence, by taking traces lie in $\mathbb{F}$. Because $\mathfrak{p}$ is solvable and π_Λ-finite, and the eigenvalues of $\pi_\Lambda(x)$ lie in $\mathbb{F}$ for any $x \in \mathfrak{p}$, $\mathfrak{p}$ must be π_Λ-triangular. Similarly, $\mathfrak{p}$ is π_Λ^*-triangular, proving *b*.

Now assume *b*. Take $\Lambda \in P_{++}$ and let $V = \pi_\Lambda(U(\mathfrak{p}))v^+$. Then $\dim V < \infty$, and there exists a $\mathfrak{p}$-invariant ideal of $\mathbb{F}[V]$ that defines the projectivization of $V \cap \mathcal{V}$. Because $V \cap \mathcal{V} \neq \emptyset$ and $\mathfrak{p}$ is triangular on V, there exists $v \in V \cap \mathcal{V}$ such that $\mathbb{F}v$ is $\pi_\Lambda(\mathfrak{p})$-invariant (cf. ref. 12). By *Theorem 1b*, choose $g_1 \in G$ such that $\pi_\Lambda(g_1)v \in \mathbb{F}^*v^+$, so that $\mathrm{Ad}(g_1)\mathfrak{p} \subset \mathfrak{b}_+$ by *Lemma 5b*. Similarly, choose $g_2 \in G$ such that $\mathrm{Ad}(g_2)\mathfrak{p} \subset \mathfrak{b}_-$. By *Corollary 2c*, write $g_1g_2^{-1} = u_+nu_-$, where $u_+ \in U_+$, $u_- \in U_-$ and $n \in N$, and put $g = u_+^{-1}g_1$. Then $\mathrm{Ad}(g)\mathfrak{p} \subset \mathfrak{b}_+ \cap \mathrm{Ad}(n)\,\mathfrak{b}_-$, proving *c*.

COROLLARY 8. *Let* $\mathfrak{p}$ *be a subalgebra of* $\mathfrak{g}$. *Then* $\mathfrak{p}$ *is* $\mathrm{ad}_{\mathfrak{g}'/\mathfrak{c}}$-*diagonalizable (resp. -triangular) if and only if there exist* $g \in G$ *and* $w \in W$ *such that* $\mathrm{Ad}(g)\mathfrak{p} \subset \mathfrak{h}$ *(resp.* $\subset \mathfrak{b}_+ \cap w(\mathfrak{b}_-)$*).*

COROLLARY 9. *Let* $\mathfrak{p}$ *be a subalgebra of* $\mathfrak{g}'$. *Then* $\mathfrak{p}$ *is* $\mathrm{ad}_{\mathfrak{g}'/\mathfrak{c}}$-*finite (resp. -triangular, -diagonalizable or -semisimple) if and only if* $\mathfrak{p}$ *is* π-*finite (resp. -triangular, -diagonalizable or -semisimple) for all integrable* $\mathfrak{g}'$-*modules* (V,π). *In particular a* $\mathfrak{g}'$-*module* (V,π) *is integrable if and only if all* $\mathrm{ad}_{\mathfrak{g}'}$-*finite elements of* $\mathfrak{g}'$ *are* π-*finite.*

COROLLARY 10. *Let* $x \in \mathfrak{g}$ *be* $\mathrm{ad}_{\mathfrak{g}}$-*finite. Then there exist unique* x_s *and* x_n *in* $\mathfrak{g}$ *such that:* $x = x_s + x_n$; $[x_s,x_n] = 0$; x_s *(resp.* x_n*) is* π_Λ- *and* π_Λ^*-*semisimple (resp. -locally nilpotent) for all* $\Lambda \in P_+$. x_s *(resp.* x_n*) is* $\mathrm{ad}_{\mathfrak{g}}$-*semisimple (resp. -locally nilpotent), and* $\mathfrak{g}^x = \mathfrak{g}^{x_s} \cap \mathfrak{g}^{x_n}$. x *is* $\mathrm{ad}_{\mathfrak{g}}$-*triangular if and only if there exist* $g \in G$ *and* $w \in W$ *such that* $\mathrm{Ad}(g)x_s \in \mathfrak{h}$ *and* $\mathrm{Ad}(g)x_n \in \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$. $x_n \in \mathfrak{g}'$ *is* π-*locally nilpotent for all integrable* $\mathfrak{g}'$-*modules* (V,π), *and* $x_n \in \mathfrak{n}_+$ *if* $x \in \mathfrak{b}_+$.

By using *Lemmas 6 and 2b*, these corollaries follow from two observations: (i) If $\mathfrak{p}$ is an abelian subspace of $\mathfrak{b}_+ \cap w(\mathfrak{b}_-)$, then there exist an abelian subspace $\mathfrak{q}$ of $\mathfrak{b}_+ \cap w(\mathfrak{b}_-)$ and a $u \in U_+ \cap wU_-w^{-1}$ such that $\mathrm{Ad}(u)\mathfrak{p} \subset \mathfrak{q} \cap \mathfrak{h} + \mathfrak{q} \cap \mathfrak{n}_+$. (ii) If x is $\mathrm{ad}_{(\mathfrak{g}'\cap\mathfrak{b}_+)/\mathfrak{c}}$-diagonalizable and $x = x_1 + x_2$, where $x_1 \in \mathfrak{h}$, $x_2 \in \mathfrak{n}_+ \cap w(\mathfrak{n}_-)$, and $[x_1,x_2] = 0$, then $x_2 = 0$.

6. Now we prove the conjugacy theorems. We call a maximal $\mathrm{ad}_{\mathfrak{p}}$-diagonalizable subalgebra of a Lie algebra $\mathfrak{p}$ a *split Cartan subalgebra.*

THEOREM 2. *Let* $\mathfrak{g} = \mathfrak{g}(A)$ *be a Kac–Moody algebra associated to a symmetrizable generalized Cartan matrix* A.

(a) *Every split Cartan subalgebra of* $\mathfrak{g}$ *(resp.* $\mathfrak{g}', \mathfrak{g}/\mathfrak{c}$, *or* $\mathfrak{g}'/\mathfrak{c}$*) is* $\mathrm{Ad}(G)$-*conjugate to* $\mathfrak{h}$ *(resp.* $\mathfrak{h}', \mathfrak{h}/\mathfrak{c}$, *or* $\mathfrak{h}'/\mathfrak{c}$*).*

(b) *If* $\mathfrak{g}_1 = \mathfrak{g}(A_1)$ *is a Kac–Moody algebra, with center* $\mathfrak{c}_1$, *such that* $\mathfrak{g}_1'/\mathfrak{c}_1$ *is isomorphic to* $\mathfrak{g}'/\mathfrak{c}$, *then* $A = A_1$ *up to a bijection of index sets.*

(c) $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{g}'/\mathfrak{c}) = \mathrm{Out}(A) \propto \mathrm{Ad}(\tilde{H} \propto G)$.

Proof: Part *a* is immediate from *Corollary 8*. Parts *b* and *c* for $\mathfrak{g}/\mathfrak{c}$ follow from part *a* and the fact that every root basis of Δ is W-conjugate to Π or $-\Pi$ (see ref. 1). Because $\mathrm{ad}_{\mathfrak{g}'/\mathfrak{c}}(\mathfrak{g}/\mathfrak{c})$ [$= \mathrm{Der}_{\mathbb{F}}(\mathfrak{g}'/\mathfrak{c})$ in the nonaffine case] is the linear span of all diagonalizable derivations of $\mathfrak{g}'/\mathfrak{c}$, we obtain parts *b* and *c*.

COROLLARY 11. $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{g}'/\mathfrak{c})$ *preserves the bilinear form* $(.|.)$ *on* $\mathfrak{g}'/\mathfrak{c}$.

We call a subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{p}$ a *completely solvable subalgebra* if it can be included in a full $\mathrm{ad}(\mathfrak{a})$-invariant flag of the space $\mathfrak{p}$ (i.e., $\mathfrak{p}$ is the union of $\mathrm{ad}(\mathfrak{a})$-invariant subspaces $\ldots \supset \mathfrak{a}_i \supset \mathfrak{a}_{i+1} \supset \ldots$ $(i \in \mathbb{Z})$ of $\mathfrak{p}$ such that $\cap_i \mathfrak{a}_i = (0)$, $\dim \mathfrak{a}_i/\mathfrak{a}_{i+1} \leq 1$, and $\mathfrak{a}_0 = \mathfrak{a}$). We call a maximal completely solvable subalgebra of a Lie algebra $\mathfrak{p}$ a *Borel subalgebra.*

THEOREM 3. *Every completely solvable subalgebra of the Kac–Moody algebra* $\mathfrak{g}$ *is contained in a Borel subalgebra. Every Borel subalgebra of* $\mathfrak{g}$ *is* $\mathrm{Ad}(G)$-*conjugate to* $\mathfrak{b}_+$ *or to* $\mathfrak{b}_-$. *Every Cartan subalgebra of* $\mathfrak{b}_+$ *is* $\mathrm{Ad}(U_+)$-*conjugate to* $\mathfrak{h}$.

Choose $h \in \mathfrak{h}$ such that $\alpha_i(h)$ is a positive integer for all $i \in I$. The eigenspace decomposition for ad h defines a $\mathbb{Z}$-gradation $U(\mathfrak{g}) = \oplus\, U(\mathfrak{g})_j$ and an associated filtration $U(\mathfrak{g})_{(n)} = \oplus_{j\geq n} U(\mathfrak{g})_j$. For $V \subset U(\mathfrak{g})$, put $V_j = V \cap U(\mathfrak{g})_j$, $V_{(j)} = V \cap U(\mathfrak{g})_{(j)}$. We call subspaces A and B of a vector space V *commensurable* if $\dim(A + B)/(A \cap B) < \infty$. The proof of *Theorem 3* is based on the following *Lemma 7*.

LEMMA 7. *Let* $\mathfrak{a}$ *be a subspace of* $\mathfrak{g}$ *such that for every* $x \in \mathfrak{g}$ *one has:* $\dim([\mathfrak{a},x] + \mathfrak{a})/\mathfrak{a} < \infty$. *Then* $\mathfrak{a}$ *is commensurable with one of:* (0), $\mathfrak{g}$, $\mathfrak{b}_+$, $\mathfrak{b}_-$.

Proof: Because $\mathfrak{a} + [h,\mathfrak{a}] = \mathfrak{a} + V$ for some finite-dimensional subspace V of $\mathfrak{g}$, $\mathfrak{a}$ and $\mathrm{ad}(U(\mathbb{F}h))\,\mathfrak{a} = \mathfrak{a} + \mathrm{ad}(U(\mathbb{F}h))V$ are commensurable. Thus, we may assume $\mathfrak{a} = \mathrm{ad}(U(\mathbb{F}h))\mathfrak{a}$, so that $\mathfrak{a} = \oplus\, \mathfrak{a}_j$. Put $T = \{j \in \mathbb{Z} | \mathfrak{a}_j \neq (0)\}$, $R = \{j \in \mathbb{Z} | [x,\mathfrak{a}_j] \not\subset \mathfrak{a}$ for some $x \in \mathfrak{h} + \Sigma_{i\in I}(\mathbb{F}e_i + \mathbb{F}f_i)\}$. Then R is finite by our assumption on $\mathfrak{a}$. Suppose that $T \cap \mathbb{Z}_+$ is infinite. Choose a positive n in T greater than $\max(R)$. If $m \geq 0$, then $\mathrm{ad}(U(\mathfrak{g})_m)\mathfrak{a}_n = \Sigma_{k\geq 0}\, \mathrm{ad}(U(\mathfrak{n}_-)_{-k}U(\mathfrak{h})U(\mathfrak{n}_+)_{m+k})\mathfrak{a}_n$ is contained in $\mathfrak{a}$ by the choice of n. But $\mathrm{ad}(U(\mathfrak{g}))\mathfrak{a}_n$ is a noncentral ideal of $\mathfrak{g}$ and so equals $\mathfrak{g}'$. Therefore, $\mathfrak{g}_{(n)} \subset \mathfrak{a}$. Similarly, if $T \cap -\mathbb{Z}_+$ is infinite, then $\omega(\mathfrak{g}_{(n')}) \subset \mathfrak{a}$ for some $n' \in \mathbb{Z}$. *Lemma 7* follows.

Proof of Theorem 3: We need three observations: (*i*) If $\dim \mathfrak{g} = \infty$ and if V_1, V_2 are subspaces of $\mathfrak{g}$ of finite codimension, then $[V_1,V_2] \cap \mathfrak{h} \neq (0)$. (*ii*) If $\dim \mathfrak{g} = \infty$ and $k,m,n \in \mathbb{Z}$, then there exists $\ell \in \mathbb{Z}$ such that $\{x \in \mathfrak{g}_{(k)} | [x,\mathfrak{g}_{(\ell)}] \subset \mathfrak{g}_{(m)}\} \subset \mathfrak{c} + \mathfrak{g}_{(n)}$. (*iii*) Let $\mathfrak{a}$ be a subalgebra of a Lie algebra $\mathfrak{p}$ such that $\mathfrak{a}$ is $\mathrm{ad}_{\mathfrak{p}/\mathfrak{a}}$-triangular, and let (V,π) be an absolutely irreducible $\mathfrak{p}$-module, all over $\mathbb{F}$. If $\mathfrak{a}$ is π-triangular over an algebraic closure of $\mathbb{F}$, then $\mathfrak{a}$ is π-triangular.

Let $\mathfrak{b}$ be a completely solvable subalgebra of $\mathfrak{g}$. We now show that $\mathfrak{b}$ is $\mathrm{Ad}(G)$-conjugate to a subalgebra of $\mathfrak{b}_+$ or $\mathfrak{b}_-$. This follows from *Lemma 6* if $\dim \mathfrak{b} < \infty$, so we assume $\dim \mathfrak{b} = \infty$ and, hence, $\dim \mathfrak{g} = \infty$. By using *i*, $\dim(\mathfrak{g}/\mathfrak{b}) = \infty$. By *Lemma 7*, $\mathfrak{b}$ is commensurable with $\mathfrak{b}_+$ or with $\mathfrak{b}_-$, say for definiteness with $\mathfrak{b}_+$. Fix $\Lambda \in P_{++}$. By using *ii* and *Lemma 5a*, we show that $\mathfrak{b}$ is π_Λ-triangular over an algebraic closure of $\mathbb{F}$. By *iii*, $\mathfrak{b}$ is π_Λ-triangular. As in the proof of *Lemma 6*, this implies that $\mathfrak{b} \subset \mathrm{Ad}(g)\mathfrak{b}_+$ for some $g \in G$. Because $\mathfrak{b}_+ \subset \mathrm{Ad}(g)\mathfrak{b}_+$ implies $g \in B_+$ (by $G = U_+NU_+$) and hence $\mathfrak{b}_+ = \mathrm{Ad}(g)\mathfrak{b}_+$, it follows in particular that $\mathfrak{b}_+$ is a Borel subalgebra of $\mathfrak{g}$.

Let $\mathfrak{a}$ be a finite-dimensional $\mathrm{ad}_{\mathfrak{b}_+}$-diagonalizable subalgebra of $\mathfrak{b}_+$. Because $\mathfrak{a}$ is $\mathrm{ad}_{\mathfrak{b}_+}$- and $\mathrm{ad}_{\mathfrak{g}/\mathfrak{b}_+}$-triangular, $\mathfrak{a}$ is $\mathrm{ad}_{\mathfrak{g}}$-triangular. Hence $\mathrm{Ad}(u)\,\mathfrak{a} \subset \mathfrak{b}_+ \cap w(\mathfrak{b}_-)$ for some $u \in U_+$ and $w \in W$ by using *Lemma 6*. By the observations following *Corollary 10*, $\mathfrak{a} \subset \mathrm{Ad}(u')\mathfrak{h}$ for some $u' \in U_+$. *Theorem 3* follows.

Remarks

(*i*) All results of this paper not explicitly involving the form $(.|.)$ can be extended to the case of an arbitrary generalized Cartan matrix.

(*ii*) One may consider "formal" completions of G and $L(\Lambda)$ and extend the results of the paper to this case. It follows from the "formal" analog of *Theorem 1b* that the orbit of the highest weight vector exhausts all formal solutions of the *KdV*-type hierarchies of Hirota bilinear equations constructed by Date–Jimbo–Kashiwara–Miwa. Furthermore, the ideal of equations satisfied by these solutions is generated by Hirota bilinear equations (cf. ref. 19).

(*iii*) In a sequel to this paper, we shall define a notion of regular function on G and prove Peter–Weyl- and Borel–Weil-type theorems (one special case is treated in ref. 13). Furthermore, for $\mathbb{F} = \mathbb{C}$ we define a Hausdorff topology on G in which G is a connected simply-connected paracompact topological group. The Iwasawa decomposition is a homeomorphism, and we have a covering of G by open charts $wU_-HU_+(w \in W)$ (see ref. 19).

By Zariski (resp. metric if $\mathbb{F} = \mathbb{C}$) topology on $L(\Lambda)$ we mean the finest topology that induces the Zariski (resp. metric) topology on finite-dimensional subspaces. Put $\mathcal{F}_\Lambda = \mathbb{P}\mathcal{V}$, $C_\lambda = \mathbb{P}\mathcal{V}(\lambda)_+$, $C^\lambda = \mathbb{P}\mathcal{V}(\lambda)_-$, $\overline{C_\lambda} = \mathbb{P}\overline{\mathcal{V}(\lambda)_+}$, $C^\lambda = \mathbb{P}\overline{\mathcal{V}(\lambda)_-}$, $d_\lambda = |\phi(\lambda)|(\lambda \in W(\Lambda))$. We deduce from *Theorem 1* that $\mathcal{F}_\Lambda$ is a disjoint union of locally closed subvarieties C_λ(resp. C^λ) that are isomorphic to $\mathbb{F}^{d_\lambda}$ (resp. have codimension d_λ), and $\overline{C}_\lambda$(resp. $\overline{C}^\lambda$) is the closure of C_λ(resp. C^λ). $\mathcal{F}_\Lambda$ in the metric topology is a CW-complex with open cells C_λ, and the singular homology of $\mathcal{F}_\Lambda$ is a free $\mathbb{Z}$-module on generators of degree $2d_\lambda$. The varieties $\mathcal{F}_\Lambda$ are called *flag varieties* of G; C_λ and C^λ (resp. $\overline{C}_\lambda$ and $\overline{C}^\lambda$) are called *finite* and *cofinite Schubert cells* (resp. *varieties*). The above decompositions of the flag varieties are called *Bruhat* and *Birkhoff decompositions*. We have a homeomorphism $\mathcal{F}_\Lambda = G/P_X$. We owe to ref. 14 the idea of studying flag varieties by using representation theory. In the affine case, the study of flag varieties goes back to Birkhoff and Bott (cf. refs. 15–17). Tits has shown recently that the flag varieties depend up to isomorphism only on the set X.

(*iv*) One can show that the Hermitian form $(x|y)_0 = -(x|\omega_0(y))$ on $\mathfrak{g}$ is positive-definite on $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Using the argument of ref. 18, one deduces that every $L(\Lambda)$ carries a positive-definite K-invariant Hermitian form. As one of the applications, one can define the moment map and prove a generalization of the Schur-Horn–Kostant–Atiyah convexity theorem. This will be discussed in a subsequent paper.

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